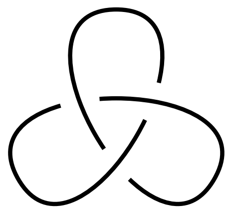
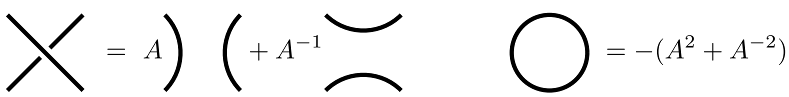
COEFFICIENTS OF QUANTUM KNOT INVARIANTS

MATTHEW HARPER

If you tie a piece of string and glue the ends together, you have created a mathematical *knot*. When thinking about knots, it is natural to ask whether a given knot can be untangled into a simple loop (an *unknot*), or whether two seemingly different knots are actually the same. Oftentimes, we will present knots using two-dimensional diagrams, such as with the trefoil below.



One way to understand knots is to assign an algebraic object to them, such as a number or a polynomial, in a way that is independent of how the knot is represented by a diagram. Such an assignment is called a *knot invariant*. One class of such invariants, inspired by quantum field theory, is called quantum invariants. They are defined by assigning matrices to each "elementary diagram" in a knot diagram, but in some cases they may be computed entirely diagrammatically. These invariants are often simple to describe but provide powerful insights into the structure and properties of knots. The most famous of these is the *Jones polynomial*, which can be computed from a knot diagram using the following rules:



The Jones polynomial of the above trefoil is $(A^2 + A^{-2})(A^5 + A^{-3} - A^{-7})$.

In this research project, we will investigate properties of a relative of the Jones polynomial, the *quantum* $\mathfrak{sl}_3$ *invariant* [Kup96, Oht02]. The plan for the project is to give an expression for the coefficients of the polynomial in terms of quantities which can be read directly from a knot diagram. Specifically, we aim to extend the results of [HK] on the set of positive alternating knots by computing the fourth and fifth coefficients of the $\mathfrak{sl}_3$ polynomial. This project can be understood from a purely combinatorial perspective; no background in topology is required.

References

[HK] Matthew Harper and Efstratia Kalfagianni. On the quantum $\mathfrak{sl}_3$ invariant of positive links. Preprint 2025, arXiv:2508.15153.

[Kup96] G. Kuperberg. Spiders for rank 2 Lie algebras. *Comm. Math. Phys.*, 180(1):109–151, 1996.

[Oht02] T. Ohtsuki. *Quantum invariants*, volume 29 of *Series on Knots and Everything*. World Scientific Publishing Co., Inc., River Edge, NJ, 2002. A study of knots, 3-manifolds, and their sets.