Recursive Polynomials

Consider a Fibonacci-type polynomial sequence given by the recurrence relation

$$G_0(x) = \alpha, G_1(x) = x + \beta, \text{ and } G_n(x) = \gamma(x)G_{n-1}(x) + G_{n-2}(x),\ n \geq 2.$$

Here α and β are integers and γ is some function of x. If $\alpha = 1,\ \beta = 0,$ and $\gamma(x) = x,$ then $G_n(x)$ is the classical Fibonacci polynomial sequence $F_n(x)$. For $\alpha = 2,\ \beta = 0,$ and $\gamma(x) = x$ one gets the Lucas polynomial sequence $L_n(x)$. Hogatt and Bicknell [3] gave explicit forms for the zeros of $F_n(x)$ and $L_n(x)$. Even though finding explicit formulas for other Fibonacci-type polynomial sequences has been found to be a challenge, several results about the properties of the zeros of $G_n(x)$ are known. For example, when $\alpha = \beta = -1,$ and $\gamma(x) = x$, G. Moore, in his 1994 paper [4] conjectured and proved that the maximum roots of the odd indexed polynomials $G_n(x)$ converge to $\frac{3}{2}$ from below and the maximum roots of the even indexed polynomials $G_n(x)$ converge to $\frac{3}{2}$ from above. In [10],Yu, Wang and He generalized Moore's result for $\alpha = \beta = a,$ and $\gamma(x) = x$ where a is any negative integer. Mátyás [5] studied the same problem for $\alpha = a, a \neq 0, \beta = \pm a,$ and $\gamma(x) = x$. We also mention the works of P. E. Ricci [9] and F. Mátyás [6] that discuss bounds for the roots of G_n. Molina and Zeleke in [7, 8] studied the asymptotic behavior of roots of G_n with $\alpha = \beta = -1,$ and $\gamma(x) = x^k$. They proved that for $k = 2$, the maximum real roots of G_{2n+1} converge to $\sqrt{2}$ from below and the maximum real roots of G_{2n} converge to $\sqrt{2}$ from above. For $k \geq 3$, they proved that the maximum roots of G_{2n+1} and G_{2n} converge from below and above respectively to the maximum root of the polynomial $x^k - x^{k-1} + x - 2$. The following are samples of projects that participants can work on.

Project 1: Find the asymptotic behavior of the roots of the k-th derivative of the Fibonacci-type polynomial sequence $G_0(x) = \alpha, G_1(x) = \beta,$ and $G_n(x) = x^k G_{n-1}(x) + G_{n-2}(x),\ n \geq 1..$ How does the order of the derivative impact the behavior of the maximum roots?

Project 2: What is the asymptotic behavior of the roots of other Fibonacci-type polynomials? For example a Fibonacci-type polynomial sequence generated using three initial conditions?

Project 3: Consider recursively defined bivariate polynomials given by

$$\begin{cases} M_n(x,y) = xM_{n-1}(x,y) + yM_{n-2}(x,y) & n \geq 2 \\ M_0 = a \\ M_1 = bx + cy + d \end{cases}$$

Finding closed forms, Binet forms and investigating the asymptotic behavior of numerical sequences at specific points in $\mathbb{R}^2$, and generating and proving sequences with combinatorial interpretation are some examples for this project.

References

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