Numerical Analysis of 1-D Parabolic Optimal Transport. This project will be on numerical analysis of a 1 dimensional, parabolic optimal transport problem. Although the optimal transport problem is a highly nonlinear and difficult problem in general, this project will deal with the greatly simplified case on the real line and has a computational component. Even in this simplified case, this direction has not been explored much, and there are still many interesting open problems.

In the optimal transport problem one has a pile of dirt and a hole of equal size, and wishes to move all of the dirt over to fill the hole. There is some cost (which might be in terms of money, energy, or some other quantity) associated to moving one unit of dirt from location x to location y, given by some function $c(x, y)$. The goal is then to move all of the dirt in a way that the total cost incurred will be minimized (i.e., an optimal way of transporting).

More rigorously, suppose that there are two domains X and $Y \subset \mathbb{R}^n$, and nonnegative Riemann integrable functions $f : X \to \mathbb{R}$, $g : Y \to \mathbb{R}$ such that $\int_X f dx = \int_Y g dy$, and a continuous function $c : X \times Y \to \mathbb{R}$. Then for any map $T : X \to Y$ for which the function $x \mapsto c(x, T(x))$ is Riemann integrable, we can define the *total cost functional*

$$\mathcal{C}_c(T) := \int_X c(x, T(x)) f(x) dx. \tag{OT}$$

The goal is then to find a mapping T which minimizes $\mathcal{C}_c$, over all maps that satisfy

$$\int_X h(T(x)) f(x) dx = \int_Y h(y) g(y) dy$$

for every continuous, bounded function $h : Y \to \mathbb{R}$ (this restriction encodes the requirement that all of the dirt is transported and there entire hole is filled up). A minimizer is called an *optimal mapping* or *Monge solution*, an optimizer will transport a mass distribution μ to ν, while minimizing the total cost measured by the integral in (OT).

This project will deal with the case when $X = [A, B]$, $Y = [C, D] \subset \mathbb{R}$, and $c(x, y) = |x - y|^2$. Then it is known that the optimal map is the derivative of some function u, where u is a convex function which satisfies the *Monge-Ampère equation*

$$\begin{aligned} \frac{d^2 u(x)}{dx^2} &= \frac{f(x)}{g(\frac{du(x)}{dx})}, \\ \frac{du(A)}{dx} &= C, \ \frac{du(B)}{dx} = D. \end{aligned} \tag{MA}$$

It is possible to look at a version of this differential equation which involves time, where one looks for a function $u : [0, \infty) \times X \to \mathbb{R}$ of variables t and x satisfying

$$\begin{aligned} &\log \frac{\partial^2 u(t, x)}{\partial x^2} - \frac{\partial u(t, x)}{\partial t} = \log \frac{f(x)}{g(\frac{\partial u(t,x)}{\partial x})}, \\ &\frac{\partial u(t, A)}{\partial x} = C, \ \frac{\partial u(t, B)}{\partial x} = D \quad \text{for all } t \geq 0, \\ &u(0, x) = u_0(x) \quad \text{for some function } u_0. \end{aligned} \tag{P-MA}$$

It is known that in this case, a solution for this *partial differential equation* exists for all times $t \geq 0$, and they converge to a solution of (MA) as $t \to \infty$.

Students will be tasked with investigating the above evolution equation using various numerical methods and for different choices of densities f and g. A few possible problems to get started are as follows:

Problem 1: What can be said about the speed of convergence to stationary solutions as $t \to \infty$?

Problem 2: Is the convergence better for certain initial conditions u_0?

Problem 3: It is possible to frame (P-MA) for more general cost functions. What happens in this case?