AFFINE MOTION OF INCOMPRESSIBLE FLUID BALLS

WILLIE WY WONG

1. Overview

The project studies the motion of a "ball" of incompressible fluid, in the restricted setting where the motion is described by affine transformations of the bodies. The goal is to answer fundamental questions about long-term behavior of solutions, especially

(1) the existence and stability of periodic orbits;
(2) the existence and non-existence of bounded orbits;
(3) classification of unbounded orbits by asymptotic behavior.

Students will learn the Euler equations describing the motion of rigid and not-so-rigid bodies, in the geometric framework as flows on Lie groups.

Students will explore explicit solutions and describe their asymptotic behavior, analytically and/or numerically.

Students will investigate theoretically the stability properties of special, periodic orbits.

2. Description

The connection between differential geometry and fluid mechanics was first realized by Arnol'd (Arnol'd, 1966), who formulated the equations of motion of incompressible inviscid fluid dynamics as geodesic motion on the infinite-dimensional Lie group of volume-preserving diffeomorphisms relative to a left-invariant metric. These Arnol'd equations are equivalent to the incompressible Euler equations. The corresponding infinite dimensional ordinary differential equation is similar in difficulty as the corresponding partial differential equation. In a recent paper (Sideris, 2017), Sideris showed that the incompressible Euler equation is compatible with the ansatz that the induced flow-map $X(t_1, t_2) : \mathbb{R}^d \to \mathbb{R}^d$ are volume preserving affine transformations. Under this ansatz the fluid equations reduce to the *geodesic equation* on the finite dimensional manifold $\mathsf{SL}(d)$ equipped with the *non*-left-invariant Riemannian metric induced by the Hilbert-Schmidt inner product on $\mathbb{M}^{d\times d}$. This Riemannian manifold is geodesically complete. A natural question to ask is: "What are the allowed asymptotic behavior of the geodesics?" Its analogue in classical celestial mechanics is the classification of two-body orbits into different conic sections. Sideris studied this question, when $d = 3$, in restricted settings. For irrotational motion, he was able to show that the orbits are necessarily unbounded, in which case asymptotically an initial fluid ball necessarily either flattens into a pancake or elongates into a cigar at late times. A similar analysis was performed for an explicit class of "shear motion" solutions which can be written down explicitly. Asymptotics for general geodesics, however, are unknown.

In this research project the students will first generalize Sideris' analysis of the irrotational and "shear motion" orbits to arbitrary dimensions, with the goal of determining whether additional types of asymptotic behavior are allowable. Additionally, it is known that when d is even certain purely-rotating solutions exist, which have no analogues when d is odd. The students will also investigate theoretically the stability properties of these periodic solutions. Finally, the students will write compute programs to numerically solve the relevant system of ordinary differential equations. The predictions of the programs will be checked against known conserved quantities–total energy and total angular momentum–as well as the special case of $d = 2$ where the motions are completely integrable. The numerical simulation will then be used to explore possible asymptotic states of the system, and to drive further analytical examinations.

References

Arnol'd, V. 1966. *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier (Grenoble) **16**, no. fasc. 1, 319–361 (French). MR0202082

Sideris, Thomas C. 2017. *Global existence and asymptotic behavior of affine motion of 3D ideal fluids surrounded by vacuum*, Arch. Ration. Mech. Anal. **225**, no. 1, 141–176, DOI 10.1007/s00205-017-1106-3. MR3634025