SPECTRAL THEORY FOR DISCRETE PERIODIC OPERATORS

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Discrete periodic operators arise from the tight-binding model in quantum mechanics, playing a crucial role in solid state theory. Before we discuss what a discrete periodic operator is, we must introduce an object essential to their definition, periodic graphs. A periodic graph is an infinite graph that is invariant under the action of a free abelian group, such as $\mathbb{Z}^d$. An example of a 2-dimensional periodic graph is the square lattice, depicted in Figure 1.

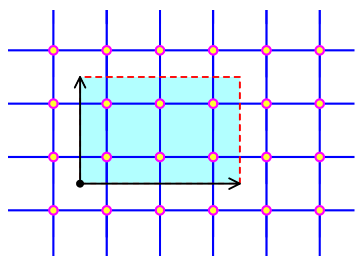

FIGURE 1. A local realization of the square lattice.

For more on periodic graphs, we point to [6].

Once given a $\mathbb{Z}^d$-periodic graph Γ, with edges $\mathcal{E}$ and vertices $\mathcal{V}$, we need a pair of functions $E : \mathcal{E} \to \mathbb{R}$ and $V : \mathcal{V} \to \mathbb{R}$ called an edge labeling and a potential, respectively. We always assume that E and V are $\mathbb{Z}^d$-periodic. A discrete periodic operator L acts on functions $f : \mathcal{V} \to \mathbb{R}$ as follows:

$$(Lf)(u) = V(u)f(u) + \sum_{(u,v)\in\mathcal{E}} E((u,v))f(v).$$

We wish to study the spectrum of L acting on the Hilbert space of square summable functions on $\mathcal{V}$. Through Floquet theory, we can reduce this problem to studying the spectrum of a finite square matrix with Laurent polynomial entries, denoted by $L(z)$. The vanishing set of the characteristic polynomial of $L(z)$ is called the Bloch variety (see Figure 2), and the vanishing set for a particular eigenvalue is called a Fermi variety.

This realization allows for the study of many properties of our operators through methods of linear algebra, combinatorics, graph theory, and algebraic geometry. Properties of interest include the irreducibility of the Bloch and Fermi varieties, isospectrality (when different labelings yield the same Bloch or Fermi variety), and more.

Much of the work on discrete periodic operators has been focused on the study of the square lattice and its higher-dimensional variants (that is, periodic graphs with the vertex set $\mathbb{Z}^d$ and nearest-neighbor edges). Questions concerning more exotic periodic graph structures have been less explored. See [2–5] for surveys that discuss much of what has

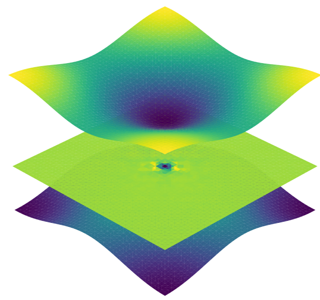

FIGURE 2. A (real) Bloch variety associated with the Lieb lattice.

been done, while noting that in the majority of these works the edge labeling is taken to be the constant function $E(\cdot) = 1$ (such operators are called discrete periodic Schrödinger operators).

Typically, when isospectrality is considered, we ask, after fixing the edge labeling, when do two Bloch varieties agree for different potentials? The alternative is largely unexplored, that is, after fixing a potential, when do the Bloch varieties of two distinct edge labelings agree. Another class of problems of interest within the study of isospectrality are Ambarzumyan-type inverse problems [1]; that is, do there exist potentials that are isospectral to the constant potential $V(\cdot) = 0$ (called the zero potential).

In this project, we will begin by introducing the mathematical background that is necessary to work with these operators. After the students have a grasp on the basics, we will utilize the computer algebra system Macaulay2 in order to investigate various spectral properties of our operators. Once we have gathered evidence, identified patterns, and are able to make conjectures, we will attempt to prove our observations.

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