PHASE RETRIEVAL ALGORITHMS IN THEORY AND PRACTICE

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Problem Motivation and Suggested Prerequisites: Students who choose this project will be introduced to both mathematical and computational aspects of the *Near-field Ptychography Problem*, an imaging problem related to the design of microscopes and other imaging systems used to "see" extremely small molecules. The mathematics involved includes, but is not limited to,

- Linear Algebra (Properties of matrices, norms, inner products, eigenvalues & eigenvectors, spectral value decomposition)
- Spectral Graph Theory (Adjacency matrices, Laplacian matrices)
- Numerical and Fourier Analysis (Angular synchronization, condition number of matrices, Complex vectors, discrete Fourier transforms, constructing algorithms)

The project is also computational in nature so programming experience will be helpful, but is not strictly necessary.

The Problem Setup: We consider a one dimensional problem for simplicity. Our specimen's image (e.g., a picture of a molecule) is represented by a vector of complex numbers $\mathbf{x} \in \mathbb{C}^d$. Our microscope that images $\mathbf{x}$ is described in terms of its Point Spread Function (PSF) $\mathbf{p} \in \mathbb{C}^d$ which, roughly speaking, describes how its lens focuses light coming from each area of the sample. Finally, we also allow ourselves the option of *masking* the sample with a partially transparent mask $\mathbf{m} \in \mathbb{C}^d$ which the microscope can shift across the specimen. This means that our microscope model is determined by three different design choices:

(1) Our PSF $\mathbf{p} \in \mathbb{C}^d$ (determining our lens)
(2) Our mask choice $\mathbf{m} \in \mathbb{C}^d$ (setting this to the vector of all ones corresponds to having "no mask"), and
(3) The set of shifts of the mask we choose to use while looking at our sample, for shifts $\mathcal{K} \subseteq \{0, 1, 2, \ldots, d-1\}$.

Given these design choices, our microscope will image the sample $\mathbf{x}$ by effectively taking a picture of its view of each shift of the mask against the sample. The picture we see is just a photon count coming from the lens' view of $\mathbf{x}$ superimposed with a shift of $\mathbf{m}$ in reality, so it corresponds to the magnitude of the information the lens (through a convolution with $\mathbf{p}$) sees from each portion of the elementwise product of a shift of $\mathbf{m}$ with $\mathbf{x}$. As a result, our imaging measurements can be modeled as a function $f_{\mathbf{p},\mathbf{m},\mathcal{K}} : \mathbb{C}^d \to \mathbb{C}^{|\mathcal{K}| \times d}$ where the $(k, \ell)^{\text{th}}$ output of $f_{\mathbf{p},\mathbf{m},\mathcal{K}}$ is given by

$$\left(f_{\mathbf{p},\mathbf{m},\mathcal{K}}(\mathbf{x})\right)_{k,\ell} := |(\mathbf{p} * (S_k\mathbf{m} \circ \mathbf{x}))_\ell|^2, \ .$$

Here, $*$ is the circular convolution of two vectors, $\circ$ is the Hadamard product of two vectors, and $S_k : \mathbb{C}^d \to \mathbb{C}^d$ is a circular shift operator defined by $(S_k\mathbf{m})_j := m_{j+k \bmod d}$. If you don't know what some of these things are, don't worry. The first days of the project will be devoted to simply understanding the basic definition of the function $f_{\mathbf{p},\mathbf{m},\mathcal{K}}$.

Our Research Questions: Having mastered the definition of $f_{\mathbf{p},\mathbf{m},\mathcal{K}}$, we will then begin to consider several questions about this type of function. These will include

- To what degree is $f_{\mathbf{p},\mathbf{m},\mathcal{K}}$ invertible (i.e., for which choices of $\mathbf{p}$, $\mathbf{m}$, and $\mathcal{K}$)?
- If $f_{\mathbf{p},\mathbf{m},\mathcal{K}}$ isn't invertible enough that we can obtain $\mathbf{x}$ (the image we want) exactly from $f_{\mathbf{p},\mathbf{m},\mathcal{K}}(\mathbf{x})$ (what we can actually measure), how much can we actually learn about $\mathbf{x}$ from $f_{\mathbf{p},\mathbf{m},\mathcal{K}}(\mathbf{x})$ for different choices of $\mathbf{p}$, $\mathbf{m}$, and $\mathcal{K}$?
- What choices of $\mathbf{p}$, $\mathbf{m}$, and $\mathcal{K}$ lead to a better microscope whose measurements $f_{\mathbf{p},\mathbf{m},\mathcal{K}}$ allow us to learn as much about $\mathbf{x}$ as we possibly can?
- What can we prove about $f_{\mathbf{p},\mathbf{m},\mathcal{K}}$ for specific types of choices of $\mathbf{p}$, $\mathbf{m}$, and $\mathcal{K}$?
- How can we numerically invert $f_{\mathbf{p},\mathbf{m},\mathcal{K}}$ to the extent it's possible to do so?