ANISOTROPIC DECAY OF NON-SCALAR FIELDS

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Introduction

Modern physics models many physical quantities in terms of *fields*, perhaps the most well-known example being the electromagnetic field modeled either as a pair $(\vec{E}, \vec{B})$ of time-dependent vector fields in space, or as a rank-2 tensor $\mathbf{F}$ over space-time. This project focuses on understanding the long-term behavior of *special relativistic*, *classical (as opposed to quantum)*, *linear* fields; this is the first step toward building and understanding models which incorporate nonlinear corrections that appear at higher energies, and also models in which different fields interact.

The fields that we will focus on "radiate". This means that for a spatially localized initial pulse, the field will begin to spread out over a wave front, and conservation of energy dictates that the field amplitude must decay. It turns out, however, the field amplitude decay is anisotropic: taking the case of electromagnetic field for example, it is known that the components of the tensor $\mathbf{F}$ exhibit a hierarchy of three different decay rates.

How are such decay captured and quantified? Two common methods used are based on, respectively, thinking of the geometric optics picture where the classical field is regarded as superpositions of traveling wave packets, and thinking of the energy picture and taking advantage of space-time symmetries. The former, in some ways, is easier to understand and compute. The latter, on the other hand, is more robust for nonlinear and interacting applications.

Project Outline

We will concentrate on four types of fields propagating on Minkowski space: the scalar field ϕ, the electromagnetic field $\mathbf{F}$, the graviton field $\mathbf{W}$, and the field χ formed from the spinor product of two scalar fields. We will use the energy approach. (If time permits, we can also look at things using the geometric optics approach.)

The decay properties of ϕ, $\mathbf{F}$, and $\mathbf{W}$ were well-understood by the early 1990 (see [K] and [CK]). In 2014 a modified approach, called the "hyperboloidal foliation method" [LM], was proposed to improve our understanding of the decay properties of ϕ.

> **Project Goal**: apply the hyperboloidal foliation method to the fields $\mathbf{F}$, $\mathbf{W}$, and the previously untreated field χ, to capture their decay properties in terms of energy integrals.

To accomplish this project, the students will learn the vector field method, a method to construct and exploit positive definite conserved quantities for hyperbolic partial differential equations, taking advantage of the symmetries of Minkowski space time. Students will upgrade the conserved quantities using the hyperboloidal foliation method to expose hidden anisotropies in the conserved energies.

Steps. The steps labeled "learn" are already known in the literature. The steps labeled "apply" will be your contribution to the literature.

- *Learn*: conservation laws for scalar fields ϕ and the vector field method
- *Learn*: hyperboloidal foliations and anisotropies
- *Learn*: conservation laws for $\mathbf{F}$ and $\mathbf{W}$
- *Apply*: hyperboloidal foliations and anisotropies for $\mathbf{F}$ and $\mathbf{W}$
- *Learn*: the weighted vector field algebra and higher order energies for ϕ
- *Apply*: higher order energies for $\mathbf{F}$ and $\mathbf{W}$
- *Apply*: anisotropies for the field χ

Preparation

Required:

- Students should be familiar with multivariable calculus, specifically being confident in the applications of the divergence theorem.

Optional (helpful, but not necessary):

- Elementary differential and Riemannian geometry of surfaces.
- Prior exposure to partial differential equations.

Bibliography

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