EIGENVALUES, CONTINUED FRACTIONS, AND THE STABILITY OF SINGULARITIES

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1. Introduction

Linear partial differential equations (PDEs) are ubiquitous in the natural sciences for modeling various physical phenomena at low energies. While these equations are fantastic to a first approximation, the linear models need to be modified by nonlinear corrections at higher energies to incorporate more complex dynamics. Nonlinear corrections allow the physical system to interact with itself which oftentimes leads to uncontrollable feedback loops and the eventual breakdown of the model. We call such a scenario the *formation of a singularity*. For example, Einstein's equations of general relativity constitute a system of nonlinear PDEs which can model the collapse of a star into a black hole (a singularity).

If these singularities are meant to model anything real, then they should be stable. For instance, if a tiny asteroid crashes into a star one second before collapsing into a black hole, we should expect our model to predict that this star still collapses into a black hole. In other words, *very small changes to the system should have at most a very small effect on the formation of a singularity*. This project will focus on quantifying the stability of singularity formation for various models from physics.

A first step in quantifying the stability of a singularity's formation is to study eigenvalues of a particular linear transformation intrinsic to that singularity. Eigenvalues are numbers which express how the physical system responds to small deviations away from the singularity. Unfortunately, proving that a specific number is an eigenvalue is a notoriously challenging problem (see [6]) because, a priori, we have no idea which of the infinitely many numbers have a chance to be eigenvalues in the first place! Fortunately, there is a numerical algorithm - *the continued fractions method* - which can provide us with evidence supporting our conjectures for those numbers which are, and are not, eigenvalues.

2. Project Outline

We will focus on four nonlinear PDEs which exhibit singularity formation:

(1) the wave maps equation,
(2) the cubic wave equation,
(3) the hyperbolic Yang-Mills equation, and
(4) the quadratic wave equation.

Using the continued fractions method as described in [8], we will compute the eigenvalues associated to the singularities that these models possess. While in some cases the eigenvalues are known (see [4, 2], [9], [3, 7], [5, 1]), it is currently unknown how the singularity exhibited by the quadratic wave equation responds to arbitrary changes.

Project Goal: use the continued fractions method to provide numerical evidence for the existence of specific eigenvalues for the quadratic wave equation.

3. Project Plan

(1) Learn how to solve ordinary differential equations (ODEs) with regular singular points via power series expansions.
(2) Learn the continued fractions method.
(3) Apply the continued fractions method to verify the known eigenvalues of
  (a) the wave maps equation,
  (b) the hyperbolic Yang-Mills equation, and
  (c) the cubic wave equation.
(4) Apply the continued fractions method to the quadratic wave equation and provide evidence for the existence of currently unknown eigenvalues.

4. Preparation

Required:

- Students should have some familiarity with solving ODEs.
- Students should have some familiarity with power series.

Useful but not required:

- Prior experience with linear algebra.
- Prior experience with Mathematica or any other computational software.

References

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